SUBJECT NAME : Discrete Mathematics

SUBJECT CODE : MA 2265

MATERIAL NAME : Problem Material

MATERIAL CODE : JM08ADM010

Name of the Student: Branch:

Unit – I (Logic and Proofs)

• Definitions

1) What are the possible truth values for an atomic statement?
2) Define a conditional statement and draw the truth table for it.
3) Express the statement, "The crop will be destroyed if there is a flood" in symbolic form.
4) Using truth table, show that $(P \to Q) \Leftrightarrow (\sim P \vee Q)$.
5) Show that $(P \wedge Q) \to (P \vee Q)$ is a tautology without using the truth table.
6) Define Bound and Free variables and give example.
7) State any two rules of inference with explanation.
8) Give the converse and the contrapositive of the implication "If it is raining, then I get wet".
9) Express the statement, "Some people who trust others are rewarded" in symbolic form.
10) What is meant by "proof by contradiction"?

• Predicate Calculus

1) Using truth table, show that $P \to (Q \to R) \Leftrightarrow P \to (Q \vee R) \Leftrightarrow (P \wedge Q) \to R$.
2) Without using truth tables, show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is tautology.
3) Without constructing the truth table show that $(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.
4) Without using truth tables, show that $(\sim P \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.
5) Show the implication without constructing the truth table, $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P)) \Rightarrow (R \to Q)$.

• Inference Theory

1) Show that S is a valid inference from the premises $P \to \sim Q,\ Q \vee R,\ \sim S \to P$ and $\sim R$.
2) Show that $R \vee S$ is a valid conclusion from the premises $C \vee D,\ (C \vee D) \to \sim H,\ \sim H \to (A \wedge \sim B)$ and $(A \vee \sim B) \to (R \vee S)$.

3) Prove that the following premises are inconsistent $P \to Q,\ Q \to S,\ R \to \sim S$ and $P \wedge R$.
4) Show that the following premises are inconsistent:
   (i) If Jack misses many classes due to illness, then he fails in high school.
   (ii) If Jack fails in high school, then he is uneducated.
   (iii) If Jack reads a lot of books, then he is not uneducated.
   (iv) Jack misses many classes due to illness and reads a lot of books.

   Hint: The given set of premises $P_1, P_2, \ldots P_n$ are said to be consistent if and only if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow T$ and the premises are said to be inconsistent if and only if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow F$.
5) Show that the following statement constitute a valid argument, by using method of derivation.

   If 'A' works hard, then either 'B' or 'C' will enjoy themselves.
   If 'B' enjoys himself, then 'A' will not work hard.
   If 'D' enjoys himself, then 'C; will not.
   Therefore, If 'A' works hard. 'D' will not enjoy himself.

6) Symbolize the following statements and then use the method of derivation. If there are meeting, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no meeting. Show that these statements constitute a valid argument.

7) Show that the following premises are inconsistent.
   (i) If Vijay misses many classes, then he fails in M.E.
   (ii) If Vijay fails in M.E., then he is unemployed.
   (iii) If Vijay appears for lot of interviews, then he is not unemployed.
   (iv) Vijay misses many classes and appears for lot of interviews.

8) Verify the validity of the inference. If one person is more successful than the other, then he has worked harder to deserve success. John has not worked harder than Peter. Therefore, John is not more successful than Peter.

- ## Quantifiers

1) Show that $(x)(P(x) \to Q(x)) \wedge (x)(Q(x)) \to R(x) \Rightarrow (x)(P(x) \to R(x))$.
2) Is the following conclusion validly derivable from the premises given? If $(\forall x)(P(x) \to Q(x)), (\exists y) P(y)$ then $(\exists z) Q(z)$.
3) Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x) P(x) \vee (\exists x) Q(x)$ by indirect method of proof.

Unit – II (Combinatorics)

• Definitions

1) State the principle of mathematical induction and strong induction.
2) Prove by induction that $n < 2^n$.
3) State and prove the pigeonhole principle.
4) State extended pigeonhole principle.
5) Define recurrence relation.
6) Write the generating function for the sequence $1, a, a^2, a^3, a^4, ...$.

• Small Problems

1) If 13 people are assembled in a room, show that atleast 2 of them must have their birthday in the same day.
2) Show that if 30 dictionaries in a library contain a total of 61327 pages, then one of the dictionaries must have atleast 2045 pages.
3) Show that $C(n,r) = C(n-1,r-1) + C(n-1,r)$.

• Mathematical Induction and Strong Induction

1) Use mathematical induction to prove that $1+2+3+...+n = \frac{n(n+1)}{2}$.
2) Use mathematical induction to prove that $1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$.
3) Using mathematical induction, prove that $1^3+2^3+3^3+...+n^3 = \left[\frac{n(n+1)}{2}\right]^2$.
4) Using mathematical induction, prove that
   $1^2+3^2+5^2+...+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.
5) Prove by mathematical induction, that $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)} = \frac{n}{n+1}$.
6) Prove that $a^n - b^n$ is a multiple of $a-b$ by using method of induction.
7) Prove that $8^n - 3^n$ is a multiple of 5 by using method of induction.
8) Using mathematical induction, prove that $3^{n+2} + 4^{2n+1}$ is divisible by 13, for all non negative integers n.
9) Use mathematical inductions to show that $n^3 - n$ is divisible by 3, for $n \in Z^+$.
10) Use mathematical induction to prove that $\left(3^n + 7^n - 2\right)$ is divisible by 8, for all $n \geq 1$.
11) State the principle of strong induction and prove that for any positive integer $n \geq 2$ is either a prime or a product of primes.

- ## Recurrence relations

1) Solve the recurrence relation $u_{n+3}-6u_{n+2}+11u_{n+1}-6u_n=0$, with $u_0=2,\ u_1=0,\ u_2=-2$.
2) Solve the recurrence relation $a_n=2\left(a_{n-1}-a_{n-2}\right)$, for $n\geq 2$ with $a_0=1, a_1=2$.
3) Solve the recurrence relation $a_{n+2}+a_n=0$, where $n\geq 0$ and $a_0=1, a_1=3$.
4) Using generating function, solve the recurrence relation $a_{n+2}-8a_{n+1}+15a_n=0$ given that $a_0=2,\ a_1=8$.
5) Use the method of generating function to solve the recurrence relation: $S(n+1)-8S(n)+16S(n-1)=4^n,\ \ n\geq 1$, with $S(0)=1$ and $S(1)=8$.
6) Solve $S(n)-2S(n-1)-3S(n-2)=0,\ \ n\geq 2$ with $S(0)=3$ and $S(1)=1$ by using generating function.
7) Solve $Y(n)-7Y(n-1)+10Y(n-2)=6+8n$ with $Y(0)=1, Y(1)=2$.
8) Solve the recurrence relation $a_n-6a_{n-1}+8a_{n-2}=3^n$ for $n\geq 2$, where $a_0=3,\ a_1=7$.
9) Solve, by using generating function, the recurrence relation $y_{n+1}-2y_n=4^n$ with $y_0=1$ for $n\geq 0$.
10) Write the recurrence relation for Fibonacci number and hence solve it.
11) Find the generating function of Fibonacci sequence $F(n)=F(n-1)+F(n-2)$ for $n\geq 2$ with for $F(0)=F(1)=1$.

- ## Inclusion and Exclusion

1) Determine the number of integers between 1 and 250 that are divisible by any of the integers 2,3,5 and 7 by principle of inclusion – exclusion.
   Hint: Let A,B,C and D be the set of integers that are divisible by 2,3,5 and 7 respectively. We have to find $\left|A\cup B\cup C\cup D\right|$.
2) Determine the number of positive integers "n" where $1\leq n\leq 100$ and n is not divisible by 2,3,5.
   Hint: Let A,B and C be the set of integers that are divisible by 2,3 and 5 respectively. First we have to calculate $\left|A\cup B\cup C\right|$ and then find $100-\left|A\cup B\cup C\right|$.
3) Determine the number of positive integers "n" where $1\leq n\leq 2000$ and n is not divisible by 2,3,5 but divisible by 7.
   Hint: Let A,B,C and D be the set of integers that are divisible by 2,3,5 and 7 respectively. We have to find $\left|\bar{A}\cap\bar{B}\cap\bar{C}\cap D\right|$. Using the formula
   $$\left|\bar{A}\cap\bar{B}\cap\bar{C}\cap D\right|=\left|\left(\overline{A\cup B\cup C}\right)\cap D\right|=\left|D\cap\left(\overline{A\cup B\cup C}\right)\right|$$
   $$=\left|D\right|-\left|D\cap\left(A\cup B\cup C\right)\right|$$ and apply Distributive property.
4) Among the 1000 +ve integers, determine the integers that are divisible by 5 but not by 7 and 9.
   Hint: Let A,B and C be the set of integers that are divisible by 5,7 and 9 respectively. We have to find $\left|A\cap\bar{B}\cap\bar{C}\right|$.

5) By using the principle of inclusion and exclusion, find the number of primes between 41 and 100?

Hint: Let A,B,C and D be the set of integers between 41 and 100 that are divisible by 2,3,5 and 7 respectively. We have to find $|A \cup B \cup C \cup D|$.

6) Use the principle of inclusion and exclusion to find the number of integer solutions of the system $x_1 + x_2 + x_3 + x_4 = 20, 1 \le x_1 \le 7,\ 1 \le x_2 \le 6,\ 5 \le x_3 \le 8, 2 \le x_4 \le 9$.

7) Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects. Find the number of students studying exactly one of the three subjects.

8) In a survey about liking colours, it was found that everyone who was surveyed had a liking for at least one of the three colours namely R, G and B. Further 30% liked Red, 40% Green and 50% Blue. Further 10% liked R &G, 5% liked G & B, 10% liked R & B. Find the percentage of surveyed people who liked all the three colours.

- ## Permutations and Combinations

1) How many different rearrangements are there of the word "REARRANGEMENT"?
2) How many solutions are there for the equation $x + y + z = 15$, where $x, y, z \ge 0$.
3) How many integer solutions are there $x + y + z = 20$, subjects to the constraints $x \ge -1,\ y \ge 0,\ z \ge 4$.

Hint: The number of integer solutions of the equation $x + y + z = d$, where $x, y, z \ge 0$, is $C(d + \text{no. of unknowns} - 1, d) = C(d + 3 - 1, d)$.(This is nCr formula)

Unit – III (Graph Theory)

- ## Definitions

1) Define (a) Graph (b) Simple graph (c) Regular graph.
2) Define a complete graph and give and example.
3) Define a pendent vertex and pendent edge.
4) Define matrix representation of a graph.
5) Define bipartite graph.
6) Define isomorphism of a graph.
7) Define complementary and self complementary graph.
8) Define connected graph and give an example for connected and disconnected graphs.
9) Find the number of connected simple graph with four vertices.
10) How many edges are there in a graph with 10 vertices each of degree 5?
11) Define vertex connectivity.
12) Define edge connectivity.
13) Define Eulerian graph.
14) Define Hamiltonian graph.
15) Define a Hamiltonian path of G.

- ## Incidence and Adjacency matrix **(Refer class note)**

- **Isomorphic verifications** **(Refer class note)**
- **Theorems**

1) Discuss the Konigsberg bridge problem.
2) Prove that in any graph G, the number of vertices of odd degree is even.
3) Prove that any self complementary graphs have 4n or 4n+1 vertices.
4) (Whitney's inequality)For any graph G, show that

   vertex connectivity $\le$ **edge connectivity** $\le$ **minimum vertex degree** and give example for above.
5) If G is k – edge connected graph then prove that $\in \ge \frac{kv}{2}$.
6) Prove that a connected graph G is Eulerian if and only if all the vertices of G are of even degree.
7) Prove that if a graph G has atmost two vertices of odd degree, then there can be Euler path in G.
8) If G is a connected graph with n vertices $n \ge 3$ and if the degree of each vertex is atleast $n/2$, then show that G is Hamiltonian.

Unit – IV (Algebraic Structures)

- **Definitions**

1) Define sub semi groups with example.
2) Give an example of semi – group but not a monoid.
3) Define semi group homomorphism.
4) Define coset and give example.
5) Define normal subgroup of a group.
6) Define kernel of a homomorphism.
7) Define ring and give example. **(or)** Define when an algebraic system $\langle S,+,.\rangle$ is called a ring.
8) Define Integral domain with example.
9) Give an example of a commutative ring without identity.
10) Define field with example.
11) Give an examples of a ring which is not a field.

- **Small Problems**

1) Let $E = \{2,4,6,8\}$. Show that $(E,+)$ and $(E,\times)$ are semi – groups but not monoids.
2) Prove that the identity element of a group G is unique.

3) Prove that the inverse element of a group G is unique.
4) State and prove the cancellation law in a group.
5) If $a \in G$, G be a group , then prove that $\left(a^{-1}\right)^{-1} = a$.
6) In a group G, $(a*b)^{-1} = b^{-1} * a^{-1} \quad \forall a,b \in G$.
7) Find the multiplication inverse of each element in Z_{11} .
8) If every element in a group is its own inverse then the group must be abelian.
9) For any $a \in G$, $a^2 = e$ then prove that G is an abelian .
10) Prove that a group can not have any element which is idempotent except the identity element.
11) Prove that a group G is abelian if and only if $(a*b)^2 = a^2 * b^2 \quad \forall a,b \in G$.
12) Find all cosets of a sub groups $H = \{1, a^2\}$ of a group $G = \{1, a, a^2, a^3\}$ under usual multiplication, where $a^4 = 1$.
13) Let $G = \{1, -1, i, -i\}$ be a group and $H = \{1, -1\}$ be a sub group of G. What is the number of distinct cosets of H in G.
14) Prove that a subgroup of an abelian group is a normal subgroup.

- **Theorems**

1) Prove that if G is an abelian group then for all $a,b \in G$ and all integer n, $(a*b)^n = b^n * a^n$.
2) If S = N X N the set of ordered pairs of positive integers with the operation * defined by $(a,b)*(c,d) = (ad+bc, bd)$ and if $f : (S,*) \to (Q,+)$ is defined by $f\left[(a,b)\right] = a/b$, then show that f is a semi – group homomorphism.
3) If S is the set of all ordered pairs (a,b) of real numbers wit the binary operation $\oplus$ defined by $(a,b) \oplus (c,d) = (a+c, b+d)$, where a,b,c,d are real, prove that $(S, \oplus)$ is a commutative group.
4) Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a*b = \frac{ab}{2}$.
5) On the set Q of all rational numbers, the operation * is defined by $a*b = a+b-ab$. Show that, under the operation *, Q is a commutative monoid.
6) Prove the necessary and sufficient condition for a non – empty sub set to be a sub group of a group.

(Or)

Prove that a non – empty subset H of a group (G,*) is a subgroup if and only if any $a,b \in H$ implies $a*b^{-1} \in H$.

7) Prove that the intersection of two normal subgroups of a groups G is also a normal subgroup of G.
8) State and prove Lagrange's theorem.
(or) Prove that the order of a subgroup of a finite group divides the order of the group.
9) Prove that the order of any element of a finite group is a divisor of the order of the group (ie) $O(a)$ is a divisor of $O(G)$ for all $a \in G$.
10) If G is a finite group, then prove that $a^{O(G)} = e$ for any element $a \in G$.
11) Prove that a subgroup H of a group G under the operation * is a normal subgroup if and only if $a^{-1} * h * a \in H$ for every $a \in G$ and $h \in H$.
12) Prove that a subgroup H of a group G is normal if and only if $xHx^{-1} = H$ for all $x \in G$.
(or) Prove that A subgroup H of G is normal if and only if left coset of H in G is equal to the right coset of H in G.
13) If N and M are normal subgroup of G, prove that NM is also a normal subgroup of G.
14) Let $(G,*)$ and (G',Δ) be groups and f is homomorphism from G to G', then prove that the kernel of f is a normal subgroup.
15) If f is a homomorphism of G onto G' with kernel K, then G/K is isomorphic to G'.
16) Prove that every finite group of order "n" is isomorphic to a permutation group of degree n. (Cayley's theorem on permutation group)

Unit – V (Lattices and Boolean algebra)

• Definitions

1) Define a poset and give an example. (or) Define partially ordered set.
2) Define a distribution lattice.
3) Give an example of a lattice which is modular but not distributive.
4) State any two properties of Lattices.
5) Define Boolean Algebra.
6) In a Boolean Algebra, prove that the complement of any element is unique.
7) Show that in any Boolean algebra, $(a+b)(a'+c) = ac + a'b + bc$.
8) Show that absorption laws are valid in a Boolean algebra.
9) Give an example of two – element Boolean Algebra.

• Lattices

1) Let N be the set of all natural numbers with the relation R as follows: $x \, R \, y$ if and only if x divides y. Show that R is a partial order relation on N.
2) Let N be set of all natural numbers and define $m \le n$ if $n - m$ is a non – negatie integer. Show that $(N,\le)$ is a poset.
3) In a Lattice $(L,\le)$, prove that $X \vee (Y \wedge Z) \le (X \vee Y) \wedge (X \vee Z)$.

4) If $(L,\le)$ is a Lattice, then for any $a,b,c\in L$ prove that $a\wedge(b\vee c)\ge(a\wedge b)\vee(a\wedge c)$.
5) Show that in a complemented distributive lattice, the De Morgan's laws hold. (or) If $(L,\vee,\wedge)$ be a complemented distributive lattice, then for any $a,b\in L$, prove (1) $\overline{a\vee b}=\overline{a}\wedge\overline{b}$ (2) $\overline{a\wedge b}=\overline{a}\vee\overline{b}$.
6) Show that in a complemented, distributed lattice, $a\le b\Leftrightarrow a\wedge\overline{b}=0\Leftrightarrow\overline{a}\vee b=1\Leftrightarrow\overline{b}\le\overline{a}$.
7) If $\langle L,\le\rangle$ be a lattice, prove the following equivalent $a,b\in L$, $a\le b\Leftrightarrow a\wedge b=a\Leftrightarrow a\vee b=b$.
8) If a,b,c are element of a distributive lattice $(L,\wedge,\vee)$ shown that $a\vee b=a\vee c$ and $a\wedge b=a\wedge c \quad\Rightarrow b=c$.
9) If $\langle L,\le\rangle$ is a ordered lattice, show that $(L,\wedge,\vee)$ is an algebraic lattice.
10) Show that every chain is a distributive lattices.
11) Show that if L is a distributive lattice than for all $a,b,c\in L$, $(a*b)\oplus(b*c)\oplus(c*a)=(a\oplus b)*(b\oplus c)*(c\oplus a)$.
12) If L is a distributive lattice with 0 and 1, show that each element has at most one complement.
13) Show that every distributive lattice is modular. Is the converse true? Justify the claim.

- ## Boolean Algebra

1) Show that in any Boolean algebra $(a+b)(a'+c)=ac+a'b+bc$.
2) Show that in any Boolean algebra, $a=b$ if and only if $ab'+a'b=0$.
3) In a Boolean algebra B prove that $(a\wedge b)'=a'\vee b'$ and $(a\vee b)'=a'\wedge b'$ for all $a,b\in B$. (DeMorgon's Law)
4) In any Boolean algebra, show that $(a+b')(b+c')(c+a')=(a'+b)(b'+c)(c'+a)$.
5) Prove that in any Boolean algebra $a\overline{b}+b\overline{c}+c\overline{a}=\overline{a}b+\overline{b}c+\overline{c}a$.
6) If x,y are elements in a Boolean algebra, then prove that $x\le y\Rightarrow x'\ge y'$.
7) If B is a Boolean algebra, then for $a\in B \quad a+1=1,\ a.0=0$.

---- *All the Best* ----